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A note on hydraulic theory of internal bores

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Abstract

Mass and momentum conservation across an internal bore, together with an assumption that energy is dissipated in both fluid layers, yields a range of possible bore speeds. The upper speed limit is that given by Wood and Simpson [Wood, I.R., Simpson, J.E., 1984. Jumps in layered miscible fluids. J. Fluid Mech., 140: 215–231.] who assume no energy dissipation in the contracting layer, while the lower limit is that of Klemp et al. [Klemp, J.B., Rotunno, R., Skamarock, W.C., 1997. On the propagation of internal bores. J. Fluid Mech., 331: 81–106] who assume no energy dissipation in the expanding layer. The two bore speeds agree to within a few percent, except when the expanding layer is shallow upstream and the internal bore propagates as a gravity current.

Keywords: Internal bores; Hydraulic jump theory; Two-layer fluid

1. Introduction

Internal bores arise in a variety of atmospheric and oceanic situations. For example, they may be generated from the interaction of atmospheric inversion layers with fronts (Clark et al., 1981). They are also a common feature in coastal oceans, where they are usually created by interaction of the tide with topographic features (e.g., Holloway, 1987). Two-layer hydraulic models are frequently applied to describe these disturbances.

Unlike the classical theory of hydraulic jumps in a single fluid layer [cf. Baines, 1995, Sec. 2.3], assuming mass and momentum conservation for an internal bore in a two-layer fluid is insufficient to close the system and to provide a relation between the bore speed and layer thicknesses. An additional closure assumption is required. Following the work of Chu and Baddour (1977), Wood and Simpson (1984, hereinafter referred to as WS) proposed that energy conservation should be assumed for the contracting

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layer, because dye-streak experiments indicate that mixing occurs primarily in the expanding layer. However, Klemp et al. (1997, hereinafter KRS) demonstrate that assuming energy conservation in the expanding layer gives better agreement with experiments for large bores advancing into a shallow layer.

In this note we show that the principles of mass and momentum conservation, coupled with an assumption that energy dissipation occurs in both layers, gives a range of possible bore speeds. The upper limit of this range is given by the WS relation and the lower limit by the KRS relation. Experimental data from the works of Baines (1984) and Rottman and Simpson (1989) generally fall within this range and support the assumption made on the energy dissipation.

2. Energy dissipation in both fluid layers

Fig. 1 shows a schematic of an internal bore in a two-layer fluid with a coordinate framework moving at the bore speed, U. Mass conservation for each layer gives

$$U_1 h_f = U h_a, \tag{1}$$

$$U_2(H - h_f) = U(H - h_a). \tag{2}$$

Momentum conservation between the upstream and downstream flows gives

$$\rho_2 U^2 (H - h_a) + \rho_1 U^2 h_a + p_l H + \frac{1}{2}\rho_2 g (H - h_a)^2 + \rho_2 g (H - h_a) h_a + \frac{1}{2}\rho_1 g h_a^2 = \rho_2 U_2^2 (H - h_f) + \rho_1 U_1^2 h_f + p_r H + \frac{1}{2}\rho_2 g (H - h_f)^2 + \rho_2 g (H - h_f) h_f + \frac{1}{2}\rho_1 g h_f^2, \tag{3}$$

where p_l and p_r denote the pressure at the upstream and downstream stations, respectively, along the upper boundary. Using Eqs. (1) and (2) and making the

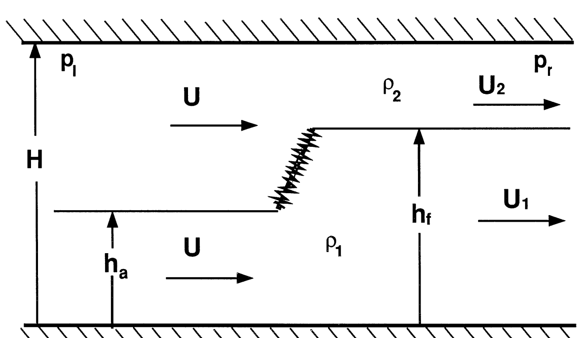

Fig. 1. Schematic of an internal bore in a two-layer fluid. The bore is assumed to be stationary in a coordinate framework moving to the left at a speed U.

Boussinesq approximation, we obtain from Eq. (3)

$$\frac{\Delta p}{\rho} = \frac{p_r - p_l}{\rho} = \frac{(h_f - h_a)}{H} U^2 \left[\frac{h_a}{h_f} - \frac{H - h_a}{H - h_f} \right] + \frac{g'}{2H}\left(h_a^2 - h_f^2\right), \tag{4}$$

where $\rho \approx \rho_1$ is a reference density and $g' = (\rho_1 - \rho_2) g/\rho$ is the reduced gravity [cf. Eq. (2) in KRS].

The net energy flux into a bore is the difference between the energy flux into it and that out of it. In a steady state, the net flux must be balanced by dissipation. The net energy flux into the bore in the upper layer is given by

$$E_u = \frac{1}{2}\rho U (H - h_a)(h_f - h_a)\left[-\frac{\left(h_f^2 - 3h_a h_f + 2h_a H\right)}{h_f (H - h_f)^2} U^2 + \frac{g'}{H}(h_f + h_a)\right]. \tag{5}$$

Assuming energy conservation in the upper layer produces

$$U_{ws} = \left\{ \frac{g' h_f (h_f + h_a)(H - h_f)^2}{H\left(h_f^2 - 3h_a h_f + 2h_a H\right)} \right\}^{1/2}, \tag{6}$$

which is the bore speed given by WS. Similarly, the net energy flux into the lower layer is given by

$$E_l = \frac{1}{2}\rho U h_a (h_f - h_a)\left\{ \frac{H(h_f + h_a) + h_f^2 - 3h_f h_a}{h_f^2 (H - h_f)} U^2 - \frac{g'}{H}\left[2H - (h_a + h_f)\right] \right\}, \tag{7}$$

and assuming energy conservation in the lower layer produces

$$U_k = \left\{ \frac{g' h_f^2 \left[2H - (h_a + h_f)\right](H - h_f)}{H\left(h_f^2 + H(h_f + h_a) - 3h_f h_a\right)} \right\}^{1/2}, \tag{8}$$

which is the bore speed given by KRS.

To nondimensionalize, let $U = (g' h_a)^{1/2} u$, $(E_u, E_l) = \rho g'^{3/2} h_a^{5/2} (e_u, e_l)$, and define $r = h_a/H$ and $R = h_f/h_a$. Then from Eqs. (5)–(8), we obtain

$$e_u = \frac{(R-1)(1-r)}{2} u \left[-\frac{R^2 r - 3Rr + 2}{R(1 - Rr)^2} u^2 + (1 + R) \right], \tag{9}$$

$$u_{ws} = \left\{ \frac{R(1+R)(1-Rr)^2}{R^2 r - 3Rr + 2} \right\}^{1/2}, \tag{10}$$

$$e_l = \frac{(R-1)}{2} u \left[\frac{R^2 r - 3Rr + R + 1}{R^2 (1 - Rr)} u^2 - (2 - r - rR) \right], \tag{11}$$

$$u_k = \left\{ \frac{R^2 \left[2 - r(1+R)\right](1 - Rr)}{R^2 r - 3Rr + R + 1} \right\}^{1/2}. \tag{12}$$

For an internal bore as shown in Fig. 1, the parameters are restricted to a physically meaningful range such that $r < 1/2$, $R > 1$ and $rR < 1$.

Following the example presented in the work of Cummins and Li (1998), Fig. 2 shows the nondimensionalized energy fluxes e_u and e_l as functions of the bore speed, u, for $r = 0.05$ and $R = 5/3$. If we make the physically plausible assumption that energy is dissipated in both layers, then the bore speed is limited to the small range where both e_u and e_l are positive. The upper limit of this range, u_{ws} (solid dot), is strikingly close to the lower limit, u_k (open circle). Based on the laboratory experiments of WS, Baines (1995, Sec. 3.5) suggested that energy dissipation may occur in both layers because viscous (and turbulent) stresses tend to dissipate energy. However, KRS raised the possibility that there might be a slight energy gain in the expanding layer for upstream-propagating bores. This issue will be discussed in Section 3. For now, we assume energy dissipation in both layers and explore the consequence of combining this hypothesis with the principles of mass and momentum conservation.

The result illustrated in Fig. 2 can be generalized to all possible combinations of layer thickness ratio, r, and jump height, R. For given values (r, R), e_u and e_l are cubic functions of u with positive nonzero roots at $u = u_{ws}$ and $u = u_k$, respectively. Because $r < 1/2$ and $(R^2 - 3R)$ reaches a minimum of $-9/4$ at $R = 3/2$, we have $r(R^2 - 3R + 2/r) > (7/4)r > 0$, and the factor multiplying u^2 in Eq. (9) is negative. Similarly in

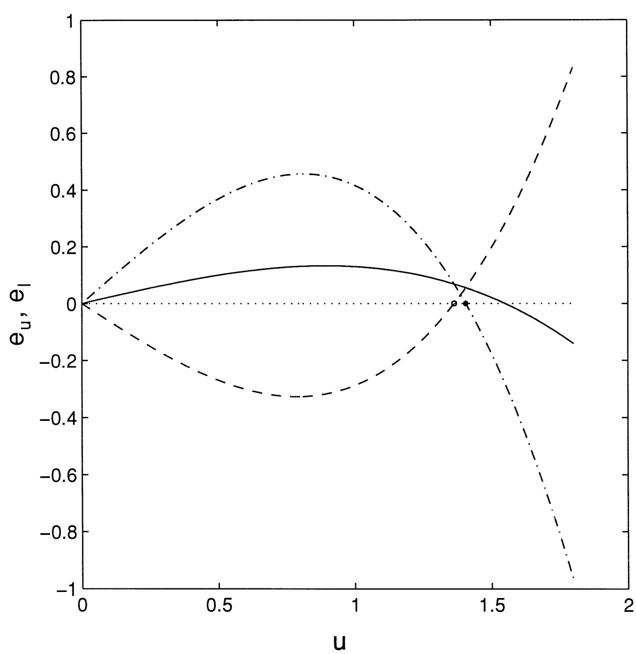

Fig. 2. Variation of the energy fluxes with the bore speed for $r = 0.05$ and $R = 5/3$. Solid line represents the total energy flux, dash–dot line the energy flux in the upper layer, e_u, and dashed line the energy flux in the lower layer, e_l. The solid dot corresponds to the bore speed given by Wood and Simpson (1984) and the open circle to that of Klemp et al. (1997).

Eq. (11), $rR^2 - 3Rr + R + 1 = R(rR + 1 - 3r) + 1 > R(1 - 2r) + 1 > 0$. Hence, for $u > 0$, e_u is convex function of u, and e_l is a concave function, as illustrated in Fig. 2. This implies that $e_u > 0$ only for $u < u_{ws}$, and $e_l > 0$ only for $u > u_k$.

It is now only necessary to show that $u_{ws} \geq u_k$. Assuming $u = u_{ws}$, Eq. (11) reduces to

$$e_l = \frac{(R-1)}{2} u_{ws} \left[\frac{R^2 r - 3Rr + R + 1}{R^2(1 - Rr)} u_{ws}^2 - (2 - r - rR) \right]. \tag{13}$$

Since $e_u(u_{ws}) = 0$ and we require no net gain of energy across the bore $(e_u + e_l) \geq 0)$, we must have $e_l \geq 0$. We then obtain from Eq. (13)

$$u_{ws}^2 \geq \frac{R^2[2 - r(1+R)](1 - Rr)}{R^2 r - 3Rr + R + 1} = u_k^2, \tag{14}$$

and hence that $u_{ws} \geq u_k$. Thus bore speeds smaller than u_k are excluded because this would imply an energy gain in the lower (expanding) layer. Bore speeds larger than u_{ws} are also excluded because an energy gain in the upper (contracting) layer would be implied. Only bore speeds between u_k and u_{ws} are consistent with the assumption of energy dissipation in both fluid layers. For $R > 1$, the total energy flux is zero $(e_u + e_l = 0)$ when $u_{ws} = u_k$; this provides an implicit relation between R and r, and determines the maximum possible bore amplitude for a given layer thickness ratio.

Fig. 3 compares u_k and u_{ws} at five different values of r for a range of R values. The two bore speeds differ by less than 7% for $r \geqslant 0.1$ and any possible value of R.

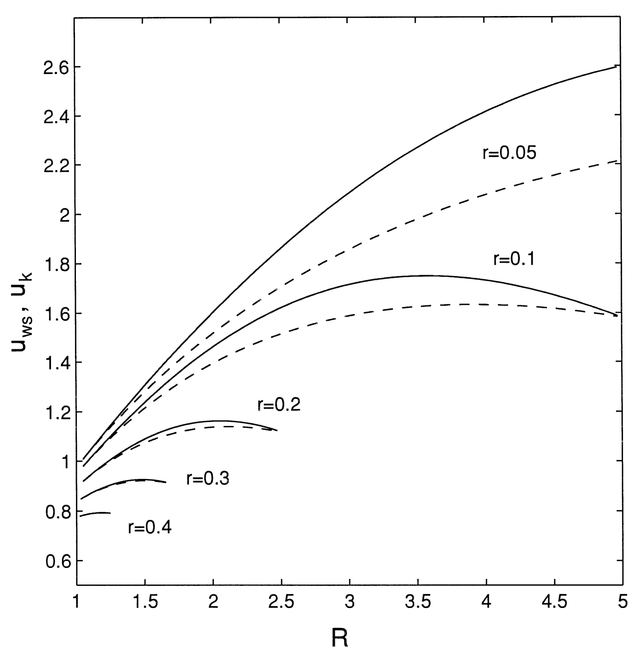

Fig. 3. Comparison of the theoretical bore speeds of Wood and Simpson (1984, solid line) and Klemp et al. (1997, dashed line) at various layer thickness ratios, r, for a range of bore amplitudes R.

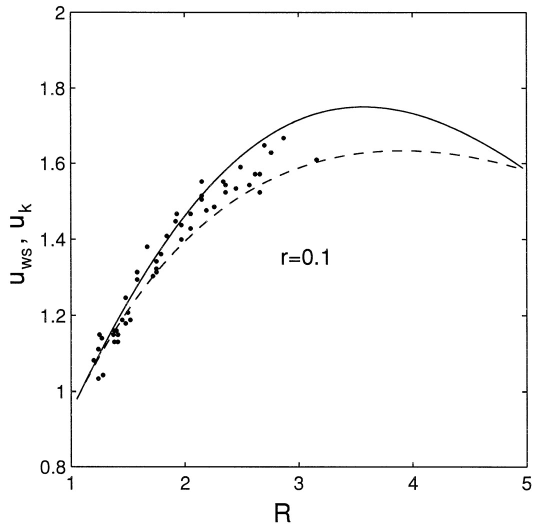

Fig. 4. Comparison between the theoretical bore speeds and the observations at $r = 0.1$. The upper solid line represents u_{ws} whereas the lower dashed line is for u_k. The experimental data (solid dots) are drawn from Fig. 12b of Baines (1984).

(The total energy flux becomes negative if R is extended beyond the right end point of each curve). Greater differences than this are obtained with $r = 0.05$ for larger values of R. Because large bores advancing into a shallow layer tend to behave like gravity currents, we expect that the KRS formula is more applicable as $r \to 0$, since it tends asymptotically to the gravity current speed of Benjamin (1968).

Fig. 4 compares the theoretical bore speeds of WS and RKS at $r = 0.1$ with experimental observations drawn from the work of Baines (1984). In agreement with our assumption on energy dissipation, the measured bore speeds generally fall between u_k and u_{ws}. Except for large bores, where their formula clearly gives better predictions, KRS obtained a similar result (e.g., their Fig. 3) in a comparison with data from Baines (1984) and Rottman and Simpson (1989) at a smaller value of r $(= 0.035)$. There is nevertheless significant scatter in the data. At small amplitudes the bores are undular and momentum conservation may only be achieved by allowing for wave resistance. [For a discussion of this matter for surface bores, see the work of Benjamin and Lighthill, 1954.]

3. Discussion

Based on the principles of mass and momentum conservation, a simple framework has been proposed to unify the hydraulic theories of internal bores. In particular we have shown that the theoretical bore speeds of WS and KRS are the upper and lower limits, respectively, of a range of bore speeds in which energy dissipation occurs in both fluid layers. Except for large bores propagating into a shallow layer, the two bore speeds agree to within a few percent, despite the different assumptions made on the underlying physics.

KRS point out that a slight energy gain may occur in the expanding layer. Based on the analysis of a model with localized turbulent stresses, they argued that eddy flux of energy between the two layers can be of either sign and may offset the dissipation term in the expanding layer. Furthermore, they present numerical simulations with a two-dimensional model that show a slight gain of energy within the expanding layer. However, the experimental data presented in Fig. 4 do not provide strong support for the hypothesis of an energy gain in the expanding layer. Assuming that the hydraulic theory is applicable, data points lying below the dashed line (u_k) imply an energy gain in the expanding layer, whereas those above the solid line (u_{ws}) imply an energy gain in the contracting layer. Except at small jump heights (R marginally greater than 1), for which it may be necessary to include wave resistance in the momentum balance, the data show little evidence for a gain of energy in the expanding layer. A greater number of data points lie between the two curves, consistent with the hypothesis that energy dissipation occurs in both layers.

The modelling studies of KRS are stimulating and call for a new set of laboratory data in which detailed quantitative measurements of mixing should be made within each constitutive layer. These measurements will help determine whether the hypothesis of energy dissipation in each layer is justifiable. An additional consideration, discussed in WS, is that the hydraulic theory of internal bores is likely to break down when significant mixing and entrainment occur between the two fluid layers, because the mass conservation Eqs. (1) and (2) will no longer be applicable.

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